5
Geostrophic Balance

http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm

The subject of this chapter is considered by Carl-Gustav Rossby and Adrian Gill to be of utmost importance

To the author's knowledge no theory exists which satisfactorily describes the mechanism whereby the mass and pressure distributions adjust themselves to the velocity distribution, although the problem is of great practical importance; its solution would be of great value not only to physical oceanographers but also to meteorologists in connection with the interpretation of the so-called dynamic pressure formations (i.e. warm anticyclones and cold cyclones).
C.-G. Rossby (1937), On the mutual adjustment of pressure and velocity distributions in certain simple current systems. *J.Marine Res*., 1, p16.

Chapter 7, perhaps the most important in the whole book, introduces effects that are due to the earth's rotation. Although Laplace included these in his tidal equations in 1778, and Kelvin investigated wave motions in a rotating fluid a hundred years later, some of the fundamental ideas were developed relatively recently by Rossby in the 1930's. The Rossby adjustment problem brings out many facets of the behavior of rotating fluids, such as the tendency to attain "geostrophic equilibrium", the significance of "potential vorticity", and the importance of the lengthscale known as the "Rossby radius of deformation".
A.E. Gill (1982), *Atmosphere-Ocean Dynamics*, Academic Press, p. xii.

5.1 The shallow water equations

In 1916 Sir Napier Shaw introduced the term "geostrophic" to designate the equilibrium between the pressure gradient force and the Coriolis force. On a scale of hundreds of kilometres or larger the atmosphere and the ocean are very frequently observed to be in approximate geostrophic equilibrium (**figure 1.77**). As far as the atmosphere is concerned, this has been known at least since Christophorus Buys Ballot deduced this from observations

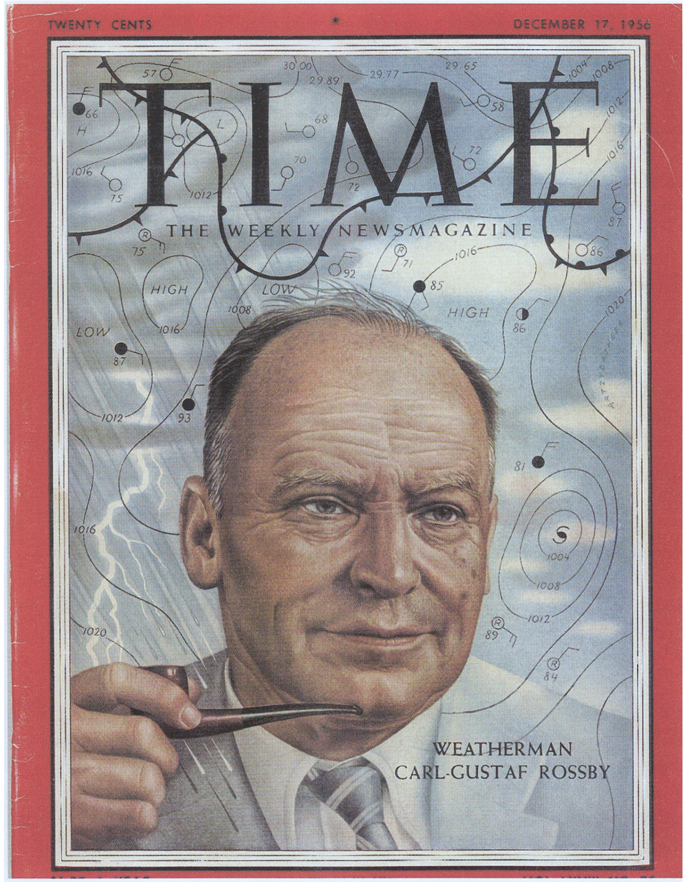

FIGURE 5.1. Carl-Gustav Rossby on the front cover of Time-magazine of 17 December 1956 as a celebrity.

in 1857 in the Netherlands[162]. Here we discuss the reasons for this remarkable fact.

The question how a rotating fluid adjusts to the state of geostrophic equilibrium or balance was addressed and answered first by Carl Gustav Rossby (**figure 5.1**) in the 1930's, surprisingly long after Buys Ballot's work was published. This chapter is inspired especially by Rossby's work, and the work by Adrian Gill (1982) (see the section, further reading, at the end of this chapter).

[162] M. Buys-Ballot, "Note sur le rapport de l'intensite et de la direction du vent avec les ecarts simultanes du barometre", *Comptes Rendus*, Vol. 45 (1857), pp. 765–768. (Buys Ballot is in general known as C.H.D. Buys Ballot. The initial M in this case stands for Monsieur).

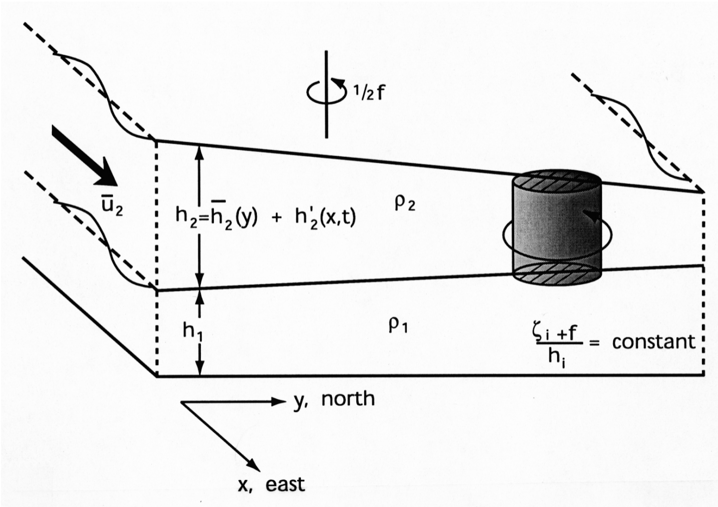

FIGURE 5.2. Schematic diagram of the two-layer model (see the text for further explanation). The basic potential vorticity gradient (south to north) is positive in the upper layer and negative in the lower layer.

Meteorologists are interested in the question how the velocity field comes into geostrophic balance with the mass- (or pressure-) field, after the mass field has been disturbed by a source of heat. Oceanographers are interested in the reverse question, that is, how the mass field adjusts to the velocity field after the velocity field has been altered by wind-stress at the ocean's surface. Here we focus in particular on the first question, adopting a **simplified layer-model of a rotating stratified fluid**, following the pioneering work of Rossby and Gill. The two-layer version of this model is illustrated in **figure 5.2**. In each layer the density is constant. This certainly appears as a rather drastic approximation, but we must keep in mind the words of John Green on "academic modelling", that are reproduced at the beginning of chapter 2. The model atmosphere shown in **figure 5.2** contains some important physical elements, namely **density-stratification and rotation**, the interplay of which we intend to investigate in this chapter. Of course, two isentropic layers (=layers of constant potential temperature) would be a better representation of the atmosphere[163]. However, we do not want to complicate matters by retaining the effects of compressibility. The 2-layer model that is illustrated in **figure 5.2** can be viewed as the "*E. Coli*" or "fruit fly" of Geophysical Fluid Dynamics. Like in Biology, where much is learned from the study of relatively very simple organisms, like bacteria, fruit flies or rats, in Geophysical Fluid Dynamics much of what we learn from the rotating constant density layer model is directly relevant to deciphering the workings of the more complex real system[164].

First we derive the equations governing the motion in each layer. We begin by **assuming hydrostatic balance**. Therefore, pressure in each layer can be written in terms of the layer

[163] See e.g. Verkley, W.T.M., 2000: On the vertical velocity in an isentropic layer. **Q.J.R.Meteorol.Soc.**, 126, 263-274.

[164] See the very interesting article by Isaac M. Held, 2005: The gap between simulation and understanding in climate modelling. **Bull.Amer.Meteorol.Soc.**, November 2005, 1609-1614.

thickness as,

$$p_1 = p_0 + \rho_2 g h_2 + \rho_1 g(h_s + h_1 - z) \equiv p_0 + \rho_1\Phi_1; \qquad (5.1a)$$

$$p_2 = p_0 + \rho_2 g(h_s + h_1 + h_2 - z) \equiv p_0 + \rho_2\Phi_2 \quad . \qquad (5.1b)$$

Here z is the height above a reference level (e.g. sea-level), p_0 is the pressure at $z=h_s+h_1+h_2$, with h_s the height of the Earth's surface relative to the reference level. The **geopotential** in each layer is defined as

$$\Phi_1 \equiv \varepsilon g h_2 + g(h_s + h_1 - z); \qquad (5.2a)$$

$$\Phi_2 \equiv g(h_s + h_1 + h_2 - z), \qquad (5.2b)$$

where

$$\varepsilon \equiv \frac{\rho_2}{\rho_1}, \text{ with } \rho_2 < \rho_1. \qquad (5.3)$$

Neglecting the effects due to the curvature of the Earth (implying among other that the Coriolis parameter, f, is constant) and neglecting the frictional force, the equations of conservation of momentum for each layer are (if p_0 is constant)

$$\frac{du_i}{dt} = fv_i - \frac{\partial\Phi_i}{\partial x}; \qquad (5.4a)$$

$$\frac{dv_i}{dt} = -fu_i - \frac{\partial\Phi_i}{\partial y}. \qquad (5.4b)$$

Here, i=1 refers to the lower layer and i=2 refers to the upper layer. Eqs 5.2a,b provide the relation between the geopotential and the thickness of each layer.

The principle of incompressibility is used to close this system of equations. Therefore,

$$\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\right) = 0$$

The left hand side of this equation can be integrated vertically over the total depth of a layer as follows:

$$\int_z^{z+h_i}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\right)dz = \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)h_i + \int_z^{z+h_i} dw_i = \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)h_i + w(z+h_i) - w(z) = \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)h_i + \frac{dh_i}{dt}.$$

Therefore,

$$\frac{dh_i}{dt} = -h_i\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right); \qquad (5.4c)$$

Eqs. 5.4a, 5.4b and 5.4c represent the "**shallow water equations**". From these equations it can be deduced that

$$\frac{d\zeta_{\text{pot i}}}{dt} = 0, \tag{5.5}$$

where

$$\boxed{\zeta_{\text{pot i}} \equiv \frac{\zeta_i + f}{h_i}} \tag{5.6}$$

is the **potential vorticity**, and

$$\zeta_i \equiv \frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}$$

is the **relative vorticity**. Eq. 5.5 implies that **potential vorticity is conserved** **for a column of fluid**.

5.2 Geostrophic adjustment, potential vorticity and the invertibility principle

Let us first illustrate some essential characteristics of adjustment to geostrophic balance in a simplified case by assuming that ρ_2=0, p_0=0, h_s=0 and neglecting derivatives with respect to y. The system reduces to one layer, which conforms to the following equations of motion and continuity

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x} + fv, \tag{5.7a}$$

$$\frac{dv}{dt} = -fu, \tag{5.7b}$$

$$\frac{dh}{dt} = -h\frac{\partial u}{\partial x}. \tag{5.7c}$$

The subscript i has been dropped.

Suppose that we *extract* a specified volume of mass from this layer. To incorporate this effect into the model we set $h=\bar{h}+\eta_0$ at t=0, where $\bar{h}$ is a constant reference height and where

$$\eta = \eta_0 \exp\left\{-\left(\frac{x - x_0}{a}\right)^2\right\}. \tag{5.8}$$

This represents a bell-shaped perturbation in height of the free surface centred at $x=x_0$ with the parameter a representing the horizontal scale and the parameter η_0 representing the maximum amplitude of the perturbation.

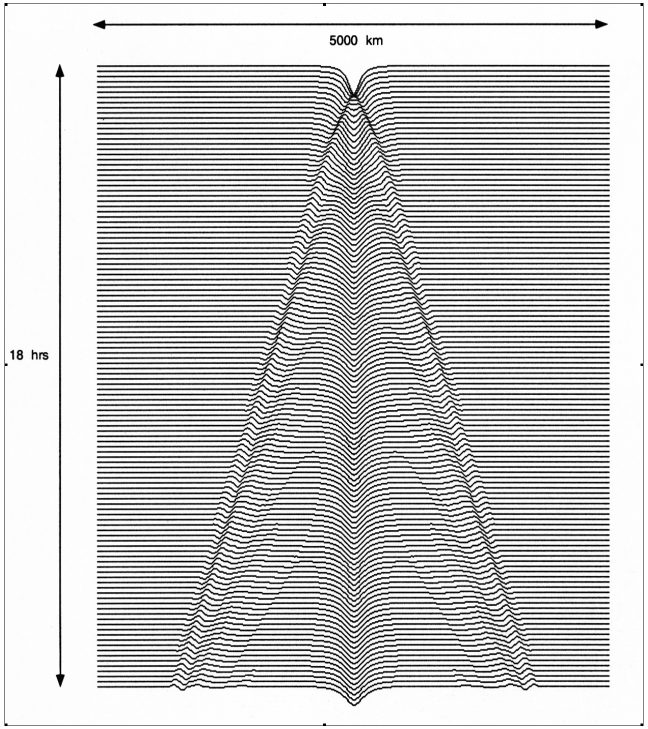

FIGURE 5.3. Height of the free surface as a function of time and horizontal distance. The initial perturbation in the free surface has a horizontal scale, a, of 60 km and a maximum amplitude of 50 m at t=0. The height, $\bar{h}$=1000 m, f=0.0005 s^{-1} and g=1 m s^{-2}. The Rossby radius is 63.2 km. The waves observed radiating away from the disturbed region are gravity-inertia waves, leaving behind in their wake the steady adjusted state.

Due to the perturbation, horizontal pressure gradients are created in the fluid leading to convergence of mass (if η_0<0) towards $x=x_0$ in the lower layer. Waves are the result. These waves propagate in both directions away from the source region around x=0. In the region around x=0 the fluid adjusts to **geostrophic balance**, here expressed by the following two equations

$$g\frac{\partial h}{\partial x} = fv;\ u = 0. \tag{5.9}$$

Figure 5.3 visualises the waves and the adjustment to geostrophic balance in the centre of the domain. The waves are called **"gravity-inertia" waves, or "Poincaré-waves"**. The dispersion relation for these waves will be derived in the following section.

Although the exact functional relation between h or v and x in the geostrophically balanced state can, in principle, be derived from eqs. 5.7 by numerical integration, it can, also be determined to a good approximation directly from (5.9) and (5.5) (conservation of potential vorticity). Potential vorticity in the one layer model with y-derivatives set to zero is defined by

$$\zeta_{pot} \equiv \frac{\zeta + f}{h} = \frac{\frac{\partial v}{\partial x} + f}{h} \qquad (5.10)$$

(relative vorticity here is $\zeta = \partial v/\partial x$). If we now differentiate (5.10) with repect to x (assuming f is constant) and use (5.9) to eliminate $\partial h/\partial x$, we obtain the following equation for v in the balanced state:

$$\boxed{\frac{d^2 v}{dx^2} - \frac{f\zeta_{pot}}{g} v = h\frac{d\zeta_{pot}}{dx}}. \qquad (5.11)$$

If ζ_{pot}>0, eq. 5.11 is a differential equation of the "**elliptic type**". Equation (5.11) is an expression of the so-called **"invertibility principle"**. In words, the invertibility principle states that the velocity distribution in the balanced state can be determined given the potential vorticity distribution and suitable boundary conditions. In the example discussed here (see **figure 5.3**) the potential vorticity in the final balanced state is, of course, not known. However, we do know that ζ_{pot} is materially conserved (eq. 5.5). If we neglect horizontal advection of ζ_{pot} (it is not obvious that we are allowed to do this, but for $\eta_0 << \bar{h}$ this is a reasonable assumption), the potential vorticity distribution at t=0 is identical to the potential vorticity distribution at any later time. We can then solve (5.11) numerically (by succesive over-relaxation; see problem 5.1) assuming that $h(t\text{->}\infty)$ on the r.h.s. of (5.11) is equal to $\bar{h}$. This yields the solution shown by the thick dashed line in **figure 5.4**, which, within a certain distance from the place of insertion of the perturbation, is nearly identical to the solution of the time-dependent eqs. 5.7a,b,c after 96 hours of integration. This remarkable fact implies that the potential vorticity, which is inserted initially, indeed stays in place. In other words, the waves hardly transport (or advect) potential vorticity away from the source region. Since potential vorticity completely determines the balanced state, this balanced state therefore must be nearly identical in both cases, in spite of the presence of waves in one case.

The solution of the homogeneous part of eq. (5.11) is of the form

$$v \propto \exp\left(\pm\frac{x}{\lambda}\right), \qquad (5.12)$$

where

$$\boxed{\lambda \equiv \sqrt{\frac{g}{f\zeta_{pot}}}} \qquad (5.13)$$

is the "**Rossby radius of deformation**", named after Carl Gustav Rossby (**figure 5.1**), who was the first to identify this imporant length scale in the 1930's (see the first quote in the beginning of this section). Solution (5.12) implies that horizontal variations in potential vorticity **force or "induce" a velocity field with a characteristic horizontal scale equal to λ**. If $\zeta << f$ the Rossby radius can be expressed as

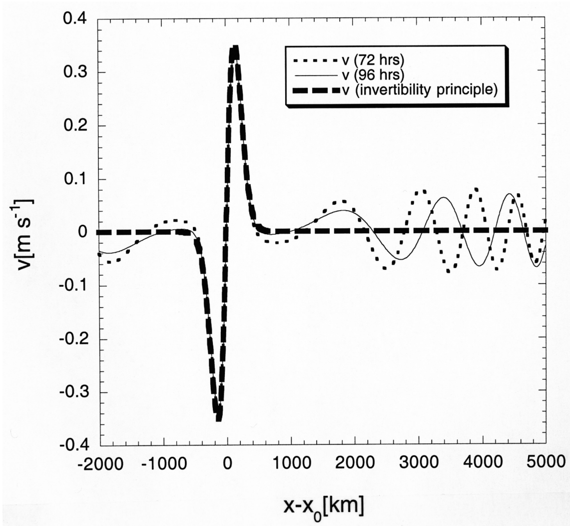

FIGURE 5.4. The velocity, v, as a function of horizontal distance according to the invertibility principle (eq. 5.11) (thick dashed line) compared to the time-dependent solution for t=72 hrs (dotted line) and t=96 hrs (thin solid line). The initial perturbation in the free surface has a horizontal scale, a, of 180 km and a maximum amplitude of 50 m at t=0. The height , $\bar{h}$ =1000 m, f=0.0005 s^{-1} and g=1 m s^{-2}. The Rossby radius is 63.2 km.

$$\lambda \equiv \frac{\sqrt{g\bar{h}}}{f}, \qquad (5.14)$$

i.e. as the ratio of the phase speed of surface gravity waves and the Coriolis-frequency.

The Rossby radius of deformation can be viewed as the analogue of the deformation height for hydrostatic adjustment (chapter 3). The Rossby radius of deformation is the e-folding distance characterizing the horizontal scale of the pressure perturbation in the centre of the domain in the final geostrophic equilibrium (**figure 5.4**).

The processes just discussed must be considered as an analogue of what happens in the real atmosphere when it is heated locally. The mass extracted is an analogue for the heating, because both effects disturb the pressure distribution as well as the potential vorticity distribution. The question how exactly heating influences the potential vorticity, will be treated in chapter12.

PROBLEM 5.1. **Numerical solution of equation 5.11**.

Solve eq 5.11 in the elliptic case for the interval $-L<x<L$ with $\partial v/\partial x$=0 at the boundaries (x=$-L$ and x=$+L$) for a prescribed perturbation given by eq. 5.8 (this determines the potential vorticity) for the parameter values given in figure 5.3. HINT: Use a method called "**successive over-relaxation**", which is simple to program. Divide the interval into subintervals of length Δx. Then v can be approximated by a set of I+1 values as $v(i\Delta x)$, i=0,1,2,...,I, where Δx=$2L/I$. Provided that Δx is sufficiently small compared to the scale on

which v varies, the I+1 grid point values should provide good approximations to v and its derivatives. Eq 5.11 can then be written in finite difference form as follows:

$$\frac{v_{i+1}+v_{i-1}-2v_i}{(\Delta x)^2} - Av_i = B_i,$$

where $v_i \equiv v(i\Delta x)$,

$$A = \frac{f\zeta_{poti}}{g} \text{ and } B_i \cong \bar{h}\frac{(\zeta_{poti+1}-\zeta_{poti-1})}{2\Delta x}, \text{with } \zeta_{poti} \equiv \zeta_{pot}(i\Delta x).$$

Starting with a "guess-field",v $(i\Delta x)$=0, a residue, R_i, is calculated according to

$$R_i = v_i - \frac{v_{i+1}+v_{i-1}-(\Delta x)^2 B_i}{2+A(\Delta x)^2}.$$

Since R_i should be equal to zero, the new "guess-value" for $v(i\Delta x)$ is

$$[v_i]_{new} = [v_i]_{old} - R_i.$$

The new "guess-value" for $v(i\Delta x)$ is substituted immediately before going to the next grid point, hence the adjective "successive" to describe this method. The process should converge within several scans of the grid. You can make the iteration more accurate by correcting h after each scan using (5.10) with the updated gridpoint values of v. Start by computing the value of the Rossby deformation radius according to (5.13) in these two cases and choosing a suitable grid point distance and domain size. Compute and plot the solution. Compare the numerical solution (qualitatively) with the analytical solution (5.12).

5.3 Gill's adjustment problem[165]

In this section we investigate the problem of geostrophic adjustment in a special case. Although Rossby is the pioneer of this problem, Adrian Gill worked out the specific case treated in this section. The shallow water equations (5.7a,b,c) with h_s=0, linearised around the rest state ($h=\bar{h}$ =constant) are, writing $h \equiv \bar{h} + \eta$, with $\eta << \bar{h}$ $h=\bar{h}+\eta$,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + fv, \tag{5.15a}$$

$$\frac{\partial v}{\partial t} = -fu, \tag{5.15b}$$

$$\frac{\partial \eta}{\partial t} = -\bar{h}\frac{\partial u}{\partial x}. \tag{5.15c}$$

The steady state solutions of the above system are

[165] See chapter 7 of Adrian E. Gill, 1982: **Atmosphere-ocean dynamics**. Academic Press, New York.

$$u = 0,\ v = 0,\ \eta = 0 \text{ (state of rest)}, \tag{5.16}$$

and

$$fv = g\frac{\partial \eta}{\partial x};\ u = 0 \text{ (state of geostrophic balance)} \tag{5.17}$$

From eqs. 5.15a-c the following equation it is derived:

$$\frac{\partial^2 \eta}{\partial t^2} - g\bar{h}\frac{\partial^2 \eta}{\partial x^2} + \bar{h}f\zeta = 0, \tag{5.18}$$

where the vertical component of the vorticity, $\zeta = \partial v/\partial x$. From eqs 5.15a-c we find the following invariant of motion (if f is constant)

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{f} - \frac{\eta}{\bar{h}}\right) = 0. \tag{5.19}$$

The dimensionless quantity $(\zeta/f - \eta/\bar{h})$ is the linearised version of the potential vorticity, ζ_{pot} (eq. 5.10). Eqs. 5.18 and 5.19 possess solutions of the form

$$\begin{aligned} \eta &= A\exp\{i(lx - \omega t)\}, \\ \zeta &= B\exp\{i(lx - \omega t)\}, \end{aligned} \tag{5.20}$$

where A and B are amplitudes, l is a wavenumbers and ω is the frequency, with

$$\omega^2 = g\bar{h}l^2 + f^2, \tag{5.21}$$

This is the **dispersion relation of small-amplitude gravity-inertia waves** in the shallow water model. The effect of rotation (non-zero f) is to make the waves dispersive, as can clearly be seen in **figure 5.3**.

PROBLEM 5.2. **Gravity-inertia waves**.
(a) Show that (5.20) is indeed a solution of eqs. 5.18 and 5.19 only if (5.21) is satisfied.
(b) Derive an expression for the phase speed of inertia-gravity waves in this model.
(c) Derive an expression for the group velocity of inertia-gravity waves in this model.
(d) In figure 5.3 we can observe gravity-inertia waves. How do you observe that these waves exhibit dispersion.
(e) What is the phase- and group-velocity of these waves?
(f) Compare this outcome with the theoretical prediction (eq. 5.21)? Discuss the differences.

If the initial state is known, the final (geostrophic) state can be found from eqs. 5.18 and 5.19. Let us assume that initially the fluid system is in rest with a perturbation in the height-field according to

$$\eta = -\eta_0 \operatorname{sgn}(x), \tag{5.22}$$

where sgn(x)=+1 if x>0 and sgn(x)=-1 if x<0 . Clearly this represents a state of imbalance. Gravity-inertia waves will be generated in the region of imbalance and propagate away. The system will ultimately adjust to balance. This will not be the state of rest, but the state of geostrophic balance! The reason for this is that the potential vorticity in the initial state is not distributed homogeneously with respect to x, and that, because of (5.19), this inhomogeneous distribution of potential vorticity must be retained at each point forever. Initially the potential vorticity distribution is given by

$$\zeta_{pot}(t=0) = \frac{\eta_0}{\bar{h}}\,\mathrm{sgn}(x)\ . \tag{5.23}$$

Due to eq. 5.19 this will give a final state for which

$$\bar{h}f\zeta = \eta f^2 + \eta_0 f^2\,\mathrm{sgn}(x)\,. \tag{5.24}$$

Substituting this into (5.18) and assuming $\partial^2\eta/\partial t^2$=0 we get

$$g\bar{h}\frac{\partial^2\eta}{\partial x^2} - f^2\eta = \eta_0 f^2\,\mathrm{sgn}(x)\,. \tag{5.25}$$

The solution of this equation is

$$\frac{\eta}{\eta_0} = -\mathrm{sgn}(x) + \mathrm{sgn}(x)\exp\left[\frac{-x\,\mathrm{sgn}(x)}{\lambda}\right] \tag{5.26}$$

where

$$\lambda = \frac{\sqrt{g\bar{h}}}{f}\,. \tag{5.27}$$

is the Rossby Radius of deformation for the case that f>>ζ. The height of the free surface, which initially is "deformed" only at x=0, is finally "deformed" over a distance of the order of the Rossby radius of deformation.

PROBLEM 5.3. **Final solution Rossby geostrophic adjustment problem**.
Derive an expression for $v(x)$ in the final geostrophic state and sketch both $h(x)$ and $v(x)$ in the final state.

PROBLEM 5.4. **Energy budget of geostrophic adjustment**
The potential energy, P, and the kinetic energy, K, per unit horizontal area for the one layer "shallow water" model are defined as follows:

$$P \equiv \frac{1}{2}\rho_1 g\eta^2\ , \tag{5.28a}$$

$$K \equiv \frac{1}{2}\rho_1\bar{h}\left(u^2+v^2\right). \tag{5.28b}$$

a) Show, using eqs (5.15a,b,c), that the total energy, (K+P), is conserved for a domain -X<x<+X, with u=0 at x=-X and at x=+X. For simplicity, neglect derivatives with respect to y.

b) Compute the potential energy per unit length (in the y-direction), ΔP that has been released during geostrophic adjustment in the example discussed in this section ("Gill's adjustment problem"), and compare this with the change in the kinetic energy. What has happened to the excess potential energy that has been released?

PROBLEM 5.5. **Rossby waves**
Gill's problem explains why the atmosphere is nearly always close the geostrophic equilibrium, for if any force tries to upset such an equilibrium, the gravitational restoring force acts to quickly restore a near geostrophic equilibrium under the constraint of potential vorticity conservation. However, the problem is actually more complicated than is suggested in the previous sections where we assumed that f is a constant. If we assume that $f(y)$, eq. 5.19 becomes

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{f} - \frac{\eta}{h}\right) = -\frac{\beta}{f}v\ , \tag{5.29}$$

where β=df/dy. We again neglect derivatives with respect to y, except in f. If, in addition, we make the so-called **geostrophic approximation**, by assuming that

$$v = \frac{g}{f}\frac{\partial\eta}{\partial x}\ \text{and therefore}\ \zeta = \frac{g}{f}\frac{\partial^2\eta}{\partial x^2}\ , \tag{5.30}$$

the equation (5.29) becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\lambda^2}\right)\eta + \beta\frac{\partial\eta}{\partial x} = 0\,, \tag{5.31}$$

where the parameter λ is the Rossby radius of deformation, defined according to (5.27). Eq. 5.31 is a simplified version of the so-called "**quasi-geostrophic**" vorticity equation (see chapter 9 for further details and implications of this approximation). Eq. 5.31 has wave-like solutions of the form

$$\eta = A\exp\{i(lx-\omega t)\} \tag{5.32}$$

Derive the associated dispersion relation (of **Rossby waves**) and compare this dispersion relation with the dispersion relation for gravity-inertia waves (5.21). What are the most remarkable differences. The interpretation is facilitated if you draw both dispersion relations in a dispersion diagram (ω as a function of l), as in figure 3.12. We will return to the quasi-geostrophic approximation and Rossby waves in chapter 9.

5.4 Energetics of adjustment

Let us introduce a spatial scale into the initial state in Gill's problem by specifying the initial state as follows:

$$\eta = \eta_0\left(\exp\left(-\frac{x}{a}\right)-1\right) \text{ if } x \geq 0 \tag{5.33a}$$

$$\eta = \eta_0\left(1-\exp\left(-\frac{x}{a}\right)\right) \text{ if } x < 0 \tag{5.33b}$$

instead of by eq. 5.22, which represents a Heaviside step function in which the height gradient is infinite at x=0. Instead, eq. 5.33 represents a smoothed Heaviside step function, where a represents the horizontal scale of the perturbation in the free surface height gradient, centred at x=0. The scale of the initial perturbation in the gradient of the height of the free surface is measured in units of the Rossby radius, λ. We call this measure (i.e. a/λ) the "**scale factor**".

We compute the initial available potential energy, P_0, by adding the contributions of each gridpoint, i.e.

$$P_0 = \frac{1}{2}\rho_1 g \sum_i \left[(\eta)_i\right]^2. \tag{5.34}$$

Here i is the index of the gridpoint. The summation in (5.34) is over all gridpoints. The potential energy, P_g, in the final balanced state and the kinetic energy, K_g, in the final balanced state is found by evaluating

$$P_g = \frac{1}{2}\rho_1 g \sum_i \left[(\eta_g)_i\right]^2. \tag{5.35a}$$

$$K_g = \frac{1}{2}\rho_1 \bar{h} \sum_i \left[(v_g)_i\right]^2. \tag{5.35b}$$

During the process of adjustment to geostrophic balance potential energy is reduced by

$$\Delta P = P_0 - P_g \tag{5.36}$$

Part of this loss of potential energy goes into energy associated with the gravity inertia waves, which radiate away from the location of the imbalance at and around x=0. The other part of this loss of potential energy goes into the kinetic energy associated with the geostrophically balanced state. The fraction, $K_g/\Delta P$, that goes into the final balanced state depends principally on the externally imposed value of the scale factor. The horizontal scale of the initial disturbance is referred to as **"dynamically large"** if its scale factor is larger than 1. An identical exercise can be performed with the Gaussian perturbation in the height of the free surface (eq. 5.8).

In **figure 5.5** we see that the balanced response to the disturbance in both cases, in terms of $K_g/\Delta P$, is different for a dynamically large disturbance than for a dynamically small disturbance. The conversion of available potential energy into geostrophic kinetic energy is most efficient when the perturbation is dynamically large. In both cases the fraction, $K_g/\Delta P$, approaches 0.5 for dynamically large perturbations [166]. The conversion into

[166] This is in agreement with the results of Middleton, J.F., 1987: Energetics of linear geostrophic adjustment. **J.Phys.Oceanogr.**, 25, 735-740.

balanced kinetic energy is very inefficient when the perturbation is dynamically small if the perturbation of the free surface has a Gaussian profile. In general, it may be stated that, when disturbances to balance are dynamically small , "the height- (or pressure-) field adjusts to the velocity field"[167], while the reverse is the case when the disturbance is dynamically large.

In the case of Gaussian perturbation, which is perhaps the best analogue of the consequence of an isolated source heat, the kinetic energy that goes into the balanced flow is maximized in an absolute sense when the initial disturbance is comparable in scale to the Rossby radius of deformation. This implies that the geostrophically balanced flow velocities are strongest when the horizontal scale of the disturbance to balance is of the order of the Rossby radius of deformation (Verify this yourself from the numerical solution of eq. 5.11 in problem 5.1).

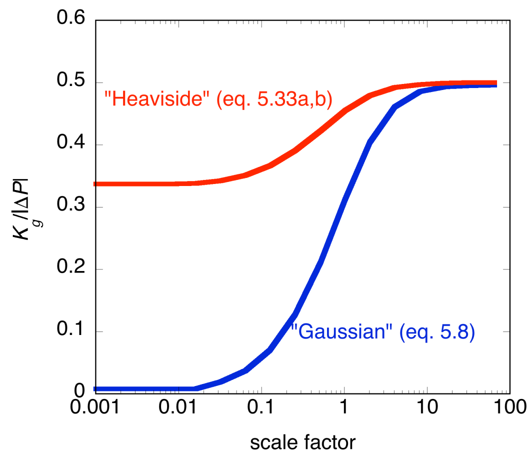

FIGURE 5.5. The geostrophic kinetic energy, K_g, in the final balanced state, relative to the absolute value of the potential energy, ΔP, that is released during adjustment, as a function of the "scale factor". The scale factor is defiend as the scale (a) of the initial perturbation relative to the Rossby radius of deformation, λ, derived from the solution of eq. 5.11. The "Gaussian" (eq. 5.8) represents an isolated perturbation in the height of the free surface. The "Heaviside" (eq. 5.33a.b) represents an isolated perturbation in the gradient of the height of the free surface. Values of other parameters are $\bar{h}$ = 1000 m , f=0.0001 s^{-1} , g=1 m s^{-2} and η_0=10 m.

5.5 Geostrophic adjustment in a two-layer model

We now formulate the invertibility principle for the two-layer model (**figure 5.2**), again neglecting derivatives with respect to y for simplicity. From the definition of potential vorticity (5.6) we can deduce that

[167] see p. 150 of G.K. Vallis, 2006: **Atmosphere and Ocean Fluid Dynamics**. Cambridge University Press,745 pp.

$$\frac{\partial^2 v_i}{\partial x^2} - \frac{\partial}{\partial x}\left\{h_i\left(\zeta_{pot}\right)_i\right\} = 0 . \tag{5.36}$$

Suppose that there is geostrophic balance, i.e.

$$\frac{\partial \Phi_i}{\partial x} = f v_i \ . \tag{5.37}$$

From eq. 5.2a,b we have

$$h_1 = \frac{\Phi_1 - \varepsilon\Phi_2}{g(1-\varepsilon)} + \left(z - h_s\right) \ , \tag{5.38a}$$

$$h_2 = \frac{\Phi_2 - \Phi_1}{g(1-\varepsilon)} \ , \tag{5.38b}$$

For the heights, h_1 and h_2 , this implies that

$$\frac{\partial h_1}{\partial x} = \frac{\frac{\partial \Phi_1}{\partial x} - \varepsilon\frac{\partial \Phi_2}{\partial x}}{g(1-\varepsilon)} - \frac{\partial h_s}{\partial x} = \frac{f\left(v_1 - \varepsilon v_2\right)}{g(1-\varepsilon)} - \frac{\partial h_s}{\partial x} \ , \tag{5.39a}$$

$$\frac{\partial h_2}{\partial x} = \frac{\frac{\partial \Phi_2}{\partial x} - \frac{\partial \Phi_1}{\partial x}}{g(1-\varepsilon)} = \frac{f\left(v_2 - v_1\right)}{g(1-\varepsilon)} \ , \tag{5.39b}$$

Substituting (5.38a,b) into (5.36) we obtain

$$\boxed{\frac{d^2 v_i}{dx^2} - A_i v_i = B_i} . \tag{5.40}$$

with

$$A_i = \frac{f\left(\zeta_{pot}\right)_i}{g(1-\varepsilon)} \ . \tag{5.41a}$$

$$B_1 = h_1\frac{\partial\left(\zeta_{pot}\right)_1}{\partial x} - \frac{\varepsilon f v_2\left(\zeta_{pot}\right)_1}{g(1-\varepsilon)} - \left(\zeta_{pot}\right)_1\frac{\partial h_s}{\partial x} \ . \tag{5.41b}$$

$$B_2 = h_2\frac{\partial\left(\zeta_{pot}\right)_2}{\partial x} - \frac{f v_1\left(\zeta_{pot}\right)_2}{g(1-\varepsilon)} \ . \tag{5.41c}$$

This is the two-layer version of the invertibility principle, analogous to eq. 5.11. It is derived from the conditions of geostrophic balance in both layers (5.37) and from the definition of potential vorticity (5.6).

Let us assume the existence of a positive potential vorticity anomaly in the upper layer, as shown in **figure 5.6**. The physical mechanisms that produce this anomaly are not addressed here. In the lower layer the potential vorticity is constant. The solution of eq. 5.40

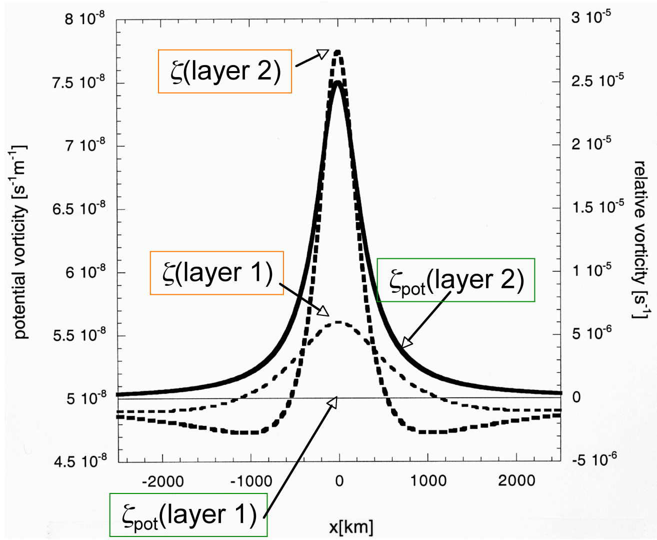

FIGURE 5.6. The potential vorticity in, respectively, the upper layer (thick solid line) and the lower layer (thin solid line) as a function of x. Also shown is the induced relative vorticity, according to the invertibility principle (eq. 5.36) in both layers (thick dashed line: upper layer; thin dashed line: lower layer). The potential vorticity anomaly is prescribed using the formula (5.8) (a bell-shaped perturbation) with a=300 km and x_0=12500 km. The static stability parameter, ε=0.8, g=9.81 m s^{-2}, f=0.0001 s^{-1} and $\overline{h_i}$=2000 m (for i=1,2).

(by the method of problem 5.1) in terms of the relative vorticity in both layers for this potential vorticity distribution with h_s=0 is also shown in **figure 5.6**. We see that the potential vorticity anomaly in the upper layer *induces* a relative vorticity anomaly in the lower layer. **The potential vorticity anomaly induces a velocity field and therefore "acts at a distance",** much in the same way as a electric charge acts at a distance by inducing an electric field. This analogy will become clearer in chapter 7.

5.6 The vacuum-cleaner effect

Let us assume that

$$\Phi_i = \overline{\Phi}_i(y) + \Phi'_i(x,t) \ . \tag{5.42a}$$

and

$$u_i = \overline{u}_i + u'_i(x,t) \ . \tag{5.42b}$$

where the time-independent state conforms to geostrophic balance, i.e.

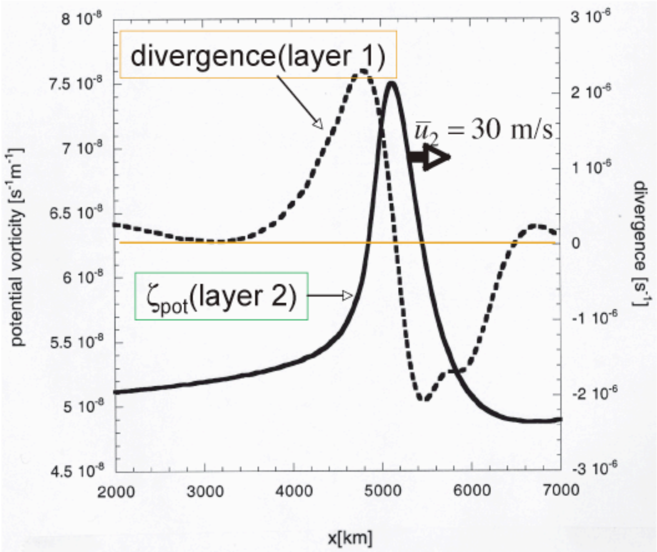

FIGURE 5.7: The numerical solution of eqs. 5.40 with $\overline{u_1}$=0 and $\overline{u_2}$=30 m/s at t=48 hours, in terms of potential vorticity in the upper layer (solid line) and divergence ($\partial u_1/\partial x$) in the lower layer (dashed line). The initial condition is a potential vorticity anomaly in the upper layer at x=0 (as shown by the thick solid line in **figure 5.6**), conforming to the invertibility principle (eq. 5.40). The amplitude of the potential vorticity anomaly remains constant, but the anomaly is deformed slightly at both the leading and trailing edge by meridional advection of "basic state" potential vorticity.

$$\frac{\partial \overline{\Phi}_i}{\partial y} = -f\overline{u}_i \; . \qquad (5.43)$$

In other words, we assume the existence of a time-independent meridional geopotential gradient, which is maintained by external forcing. The associated pressure gradient force is in balance with Coriolis force associated with a zonal flow.

The equations governing the dynamics of the two layers are (eq. 5.4a-c)

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = fv_i - \frac{\partial \phi_i}{\partial x}, \qquad (5.44a)$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = f\left(\overline{u}_i - u_i\right), \qquad (5.44b)$$

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = -h_i\frac{\partial u_i}{\partial x}, \qquad (5.44c)$$

If we now set the potential vorticity anomaly in the upper layer into motion by assuming that $\overline{u_2}$=30 m s^{-1} ($\overline{u_1}$=0), the potential vorticity anomaly will travel eastward (remember that potential vorticity is materially conserved), deforming slightly due to meridional advection

of potential vorticity associated with the meridional gradient in the basic state thickness. If the system adjusts to geostrophic balance in both layers continuously, the induced vorticity in the lower layer will necessarily also travel eastward. **The associated process of adjustment in the lower layer requires convergence in advance of the moving vorticity anomaly (where $\partial\zeta/\partial t>0$) and divergence behind the moving vorticity anomaly (where $\partial\zeta/\partial t<0$).** This is illustrated in **figure 5.7**. Hoskins et al. (1985)[168] have come up with the following instructive analogy of this process:

> One may think of an eastward-moving upper air [potential vorticity] anomaly as acting on the underlying layers of the atmosphere somewhat like a broad very gentle "vacuum cleaner", sucking air upwards towards its leading portion and pushing it downwards over the trailing portion. The vertical motion field arises in response to the need to maintain mass conservation and approximate balance. ... If a potential vorticity anomaly were to arrive overhead without any adjustment taking place underneath it, then the wind, temperature and pressure fields would be out of balance to an improbable extent.

The induced upward motion in advance of the approaching anomaly will very likely generate precipitating clouds, thereby generating potential vorticity in the lower layers of the atmosphere. The consequences of this fact will hopefully become clearer after further study (see especially section 5.9).

5.7 Complexities

The problem of geostrophic adjustment becomes much more complex when we try to make it more realistic. One complicating aspect of the problem arrises when we explicitly impose side-boundaries. In the words of Geoffrey Vallis: "in a finite domain, unless viscosity is introduced, gravity waves will forever "slosh" without dissipating"[169]. What is the fate of the waves that are generated by the adjustment process? All we can say at this moment, relating to the answer to this question, is that it appears from the numerical experiments (section 5.2), that the waves do not, or hardly, affect the potential vorticity, and therefore do not, or hardly, determine the geostrophically balanced state (**figure 5.4**), which is by no means a trivial statement.

When boundary conditions are inhomogeneous (for example, when $h_s(x)\neq 0$) we find that the qualitative interpretation of the relation between the balanced flow $v(x)$ and the potential vorticity $\zeta_{pot}(x)$ is less straightforward than is implied by the discussion in the previous sections. If $h_s(x)\neq 0$, the balanced state of rest in the single layer shallow water model (section 5.2) corresponds to $h_s(x)$ $+h(x)$=constant. Therefore, if $h_s(x)\neq 0$, then also $h(x)\neq 0$, which implies that $\zeta_{pot}(x)$ is not constant. Therefore, **with inhomogeneous boundary conditions, an anomaly in potential vorticity does not necessarily imply an anomaly in vorticity**. In the case of a fluid of constant density at rest over a mountain, which of course is a balanced state, we have a positive potential vorticity anomaly over the mountain associated with zero relative vorticity. It has been shown first by Bretherton

168 Hoskins, B.J., M.E. McIntyre and A.W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. **Q.J.R.Meteorol.Soc.**, 111, 877-946 (see page 907)

169 Vallis, G.K., 1992: Mechanisms and parameterization of geostrophic adjustment and a variational approach to balanced flow. **J.Atmos.Sci.**, 49, 1144-1160.

(1966)[170] that an inhomogeneous boundary condition can be replaced by a homogeneous boundary condition if this simplification is "compensated" by including an appropriate **imaginary potential vorticity anomaly** at the boundary. In the simple example of a layer of constant density at rest over an isolated mountain, we would interprete the mountain as a negative potential vorticity anomaly located exactly at the earth's surface and inducing a negative relative vorticity anomaly, which exactly compensates the positive relative vorticity anomaly induced by the positive potential vorticity anomaly in the interior of the fluid. Some details of the problem associated with inhomogeneous boundary conditions will be treated further in chapter 9.

Another complicating aspect arrises when we relax the assumption of constant Coriolis parameter. In fact when applying the model to the tropics we must take the south-north (meridional) variation of f into account by assuming that

$$f = 2\Omega \sin\phi \cong \beta y \ . \tag{5.45}$$

with

$$\beta = \frac{2\Omega}{a}\cos\phi \ . \tag{5.46}$$

Sometimes it is assumed that β is constant. This is referred to as the **β-plane approximation**. It is the most simple method to incorporate the gross (first order) effects of the sphericity of the Earth. Things become more difficult, but no less interesting, when account is taken of the *all* the subtle inertial effects of the sphericity of the Earth[171].

A third source of complexity is related to the possibility of **inertial instability** (section 1.20) in which case geostrophic adjustment is not possible without strongly non-linear effects that are hard to capture in a simple model.

170 Bretherton, F.P., 1966: Critical layer instability in baroclinic flows. **Quart.J.Roy.Meteorol.Soc.**, 92, 325-334.

171 See e.g. Gerkema, T., J.T.F. Zimmerman, L.R.M. Maas and H. van Haren, 2008, Geophysical and Astrophysical Fluid Dynamics beyond the traditional approximation. **Rev.Geophysics, 46**, RG2004, doi:10.1029/2006RG000220.

ABSTRACT OF CHAPTER 5

Chapter 5 is concerned with the process of **adjustment to geostrophic balance**. A simplified **hydrostatic model** of the atmosphere (one homogeneous layer) is employed to illustrate the fundamental characteristics of this process. Material **conservation of potential vorticity** is demonstrated for this model. It is shown that (in this simplified model) the role of **gravity-inertia waves** in the process of adjustment to geostrophic balance is analogous to the role of sound waves in the process of adjustment to hydrostatic balance (chapter 3). The velocity distribution in the geostrophically balanced state is directly related to the potential vorticity distribution by the **invertibility Principle**, which takes the mathematical form of a nonlinear **elliptic differential equation**. The linearized version of this equation can easily be solved numerically (given suitable boundary conditions) by successive relaxation. The analytic solution of the homogeneous part of this equation can be obtained analytically. From this solution it appears that, in the state of geostrophic balance, a positive (negative) **potential vorticity anomaly** is associated with a cyclonic (anticyclonic) wind distribution with a horizontal scale in the order of two times the **Rossby radius of deformation**. This scale is the analogue of the vertical "scale height" associated with adjustment to hydrostatic balance (chapter 3). A discussion of the energetics of geostrophic adjustment further illustrates the importance of the **Rossby radius of deformation as a horizontal scale separating dynamically large scale flow from dynamically small scale flow**. The response to a disturbance in the mass- or pressure-field that is dynamically large releases very little potential energy and of the potential energy that is relaesed relatively more is converted to kinetic energy.

In the latter part of chapter 5 a two-layer model is used to illustrate the concept of **action at a distance** in the vertical direction of a potential vorticity anomaly and the **vacuum cleaner effect** of a moving potential vorticity anomaly. Some of these topics are discussed further in chapter 7.

Geostrophic adjustment in the equatorial area, where the Coriolis parameter varies from positive to negative values with latitude is not discussed in chapter 5. It will be the subject of chapter 13.

Further reading

Books

Gill, A.E., 1982: **Atmosphere-Ocean Dynamics**. Academic Press, 662 pp. (Chapter 7).

Holton, J.R., 2004: **An Introduction to Dynamic Meteorology**. Academic Press, 529 pp. (see p. 208-213).

Vallis, G,K., 2006: **Atmospheric and Ocean Fluid Dynamics**. Cambridge University Press, 745 pp. (see p. 144-152).

Mak, M., 2011: **Atmospheric Dynamics**. Cambridge University Press. 486 pp. (Chapter 7)

More specialist articles

E. Boss and L. Thompson, 1995: Energetics of nonlinear geostrophic adjustment. **J.Phys.Oceanogr**., 25, 1521-1529. (Further details on the subject of section 4.4).

Blumen, W., 1972: Geostrophic adjustment. **Reviews of Geophysics and Space Physics**., 10, 485-528. (A comprehensive review of geostrophic adjusment that exists).

Eliassen, A., 1984: Geostrophy. **Quart.J.R.Meteorol.Soc.,** 110, 1-12. (Written by one of the greatest theoreticians in dynamical meteorology of the twentieth century)

Middleton, J.F., 1987: Energetics of linear geostrophic adjustment. **J.Phys.Oceanogr**., 25, 735-740. (Further details on the subject of section 5.4).

List of problems (chapter 5)

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http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm